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Chandrasekar Equations for Infinite Dimensional Systems:

Part II. Unbounded Input and Output Case

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Abstract

A set of equations known as Chandrasekhar equations arising in the linear quadratic optimal control problem is considered. In this paper, we consider the linear time-invariant systems defined in Hilbert spaces involving unbounded input and output operators. For a general class of such systems, we derive the Chandrasekhar equations and establish the existence, uniqueness, and regularity results of their solutions.

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1. Introduction

During the last two decades, there has been an extensive literature concerning linear quadratic regulator (LQR) problems for infinite dimensional sy stems which involves unbounded input operator in the evolution equation and/or unbounded output operator in the quadratic cost functional (see [1], [5], [17], [19], [22], [23], and [25] and the references cited there, for surveys of the recent results). The optimal control to LQR-problem is given by a feedback form involving the solution of Riccati equations. Thus, the main issue in this subject has been the study of existence and uniqueness of solutions of Riccati equations. The paper by Banks and Burns [2] followed by Gibson's result [9] have addressed the computational aspects of LQR problem for infinite dimensional systems using the approximation results of semigroups.

This paper intends to develop an alternative approach based on Chandrasekhar-type equations [4], [15]. In [13], we have considered LQR problem for systems with bounded input and output operators and derived the Chandrasekhar equations for optimal feedback gain operators. Moreover, the form of the Chandrasekhar equation allowed us to obtain differentiability results for solutions to the associated Riccati equation and the optimal control in time.

The purpose of this paper is to extend the results in [13] to systems with unbounded input and output operators. Recently, Pritchard and Salamon [22] have introduced a framework based upon semigroup theory for LQR problems involving unbounded input and output operators, which we shall describe in Section 2. Within the framework in Section 2, we show the existence, uniqueness, and differentiability results for solutions of the Chandrasekhar equation in Section 3. A number of examples which can be handled by the results in Section 3 are discussed in Section 4. In Section 5, we state the corresponding results for an important class of problems which cannot be covered by the main result; e.g., the evolution system with delays in control and the parabolic and hyperbolic systems with Dirichlet boundary control.

The computational aspects of the Chandrasekhar algorithm have been studied in [3] where the input and output operators are bounded. An extension of such a study for unbounded operator case will be reported in the forthcoming paper.

Throughout this paper, the symbol (') will be used to denote dual operators and dual spaces [28] and the dymbol (*) will denote the Hilbert space adjoint. For Hilbert spaces X and Y, we shall denote by $C_s(a,b;\mathfrak{L}(X,Y))$, the set of all mapping $t \longrightarrow F(t) \in \mathfrak{L}(X,Y)$ on $[a,b]$ such that $F(t)x$ is strongly continuous for any $x \in X$.

2. A Basic Framework for Systems with Unbounded Input and Output Operators

Assume H , U , and Y are Hilbert spaces, and we identify them with their duals. In a formal sense, our basic model [10], [25] is

$$\frac{d}{dt}x(t) = \mathcal{A}x(t) + \mathcal{B}u(t)\,, \qquad x(0) = x_0$$

(2.1)

$$y(t) = \mathcal{C}x(t)$$

where $u \in L_2(0,T;U)$, $y \in L_2(0,T;Y)$. $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on the Hilbert space H with domain $D(\mathcal{A}) \subset H$. Here,

$$\mathcal{B}U \subset D(\mathcal{A}^*)' \quad \text{and} \quad D(\mathcal{A}) \subset D(\mathcal{C})$$

where $D(\mathcal{A}^*)$ is the Hilbert space equipped with graph norm and $D(\mathcal{A}^*) \subset H \subset D(\mathcal{A}^*)'$. We interpret equation (2.1) in the mild sense: the solution of (2.1) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\mathcal{B}u(s)\,ds \tag{2.2}$$

Since $S(t)$ can be extended as a strongly continuous semigroup on $D(\mathcal{A}^*)'$ [14], [24], $x(t)$ is a $D(\mathcal{A}^*)'$ -valued continuous function.

Moreover, as in [22], we assume the following to discuss the problem involving possible unboundedness of the operators $\mathcal{B}$ and $\mathcal{C}$: $\mathcal{B} \in \mathfrak{L}(U,V)$ and $\mathcal{C} \in \mathfrak{L}(W,Y)$ where W and V are Hilbert spaces such that

$$W \subset H \subset V$$

with continuous dense injections $k: W \longrightarrow H$ and $\ell: H \longrightarrow V$. In order to make the expression (2.2) precise and to allow for trajectories in all three spaces W, H and V, we assume the following hypothesis:

(H1) $S(t)$ *is also strongly continuous semigroup on* W *and* V, *which means that there exists strongly continuous semigroups* $S_W(t)$ *and* $S_V(t)$ *and* W *and* V, *respectively, satisfying*

$$S(t)kx = kS_W(t)x \qquad \text{for } x \in W$$

and

$$S_V(t)\ell x = \ell S(t)x \qquad \text{for } x \in V .$$

Thus, if $i = \ell k$, *the continuous dense injection from* W *into* V, *then*

$$iA_W x = A_V ix \qquad \text{for } x \in D_W(A_W) = \{x \in W,\ A_W x \in W\} .$$

The subscript for the underlying Hilbert space will be omitted when understood from the context.

(H2) *For any* $u \in L_2(0,T;U)$

$$\int_0^T S(T-s)Bu(s)ds \in i(W)$$

and there exists a positive constant b *such that*

$$\left\| i^{-1} \int_0^T S(T-s)Bu(s)ds \right\|_W \leqslant b\ \|u\|_{L_2(0,T;U)} .$$

(H3) *There exists a positive constant* c *such that*

$$\|CS(t)x\|_{L_2(0,T;Y)} \leq c\|ix\|_V \qquad \text{for } x \in W \ .$$

(H4) *Suppose* $Z = D_V(A) \subset W$ *with a continuous dense embedding where* Z *is the Hilbert space* $D_V(A)$ *with the graph norm of* A_V *on* V .

Remark. It has not been explicitly stated, but each of the embedding maps is an element into itself in the larger space. For example, if $x \in W$, then $ix = x \in V$. It follows from (H4) that $D_V(A)$ is in the range of i.

By duality

$$V' \subset H = H' \subset W'$$

with continuous dense embeddings [24]. Moreover, $S'(t)$ is a strongly continuous semigroup on all three spaces V', H, W' [28, p. 273]. The following duality results will play an important role.

Theorem 2.1. *The dual statements of* (H2) *and* (H3) *are given by*

(H2)' *for every* $x \in V'$

$$\|B'S'(T-\cdot)x\|_{L_2(0,T;U)} \leq b\|i'x\|_{W'} \ ,$$

(H3)' *for every* $y(\cdot) \in L_2(0,T;Y)$

$$\int_0^T S'(T-s)C'y(s)\,ds \in i'(V')$$

and

$$\left\|(i')^{-1} \int_0^T S'(T-s)\,C'y(s)ds\right\|_{V'} \leq c\|y\|_{L_2(0,T;Y)} \ .$$

Proof. (H2) implies that for every $u \in L_2(0,T;U)$ there exists a $z \in W$ such that

$$iz = \int_0^T S(T-s)\mathcal{B}u(s)\,ds$$

and

$$\|z\|_W \leq b\|u\|_{L_2(0,T;U)} \,.$$

For $x \in V'$

$$\langle iz,x\rangle_{V,V'} = \langle\int_0^T S(T-s)\mathcal{B}u(s)\,ds,x\rangle_{V,V'}$$
$$= \int_0^T \langle\mathcal{B}'S'(T-s)x,u(s)\rangle_U ds \,.$$

But since

$$|\langle iz,x\rangle_{V,V'}| = |\langle z,i'x\rangle_{W,W'}| \leq \|z\|_W \|i'x\|_{W'} \,,$$

letting $u = \mathcal{B}'S'(T-\cdot)x \in L_2(0,T;U)$, we obtain

$$\int_0^T \|\mathcal{B}'S'(T-s)x\|_U^2 \leq h\|u\|_{L_2(0,T;U)} \|ix\|_{W'}$$

which shows (H2)'.

Next, we shall show (H3) $\Rightarrow$ (H3)' . Let $y \in L_2(0,T;Y)$ and $x \in W$. Then

$$\langle y, CS(T-\cdot)x\rangle_{L_2(0,T;Y)}$$

$$= \int_0^T \langle y(s), CS(T-s)x\rangle_Y \, ds$$

$$= \int^T \langle S'(T-s)C'y(s), x\rangle_{W',W} \, ds$$

$$= \langle \int_0^T S'(T-s)C'y(s)\, ds, x\rangle_{W',W} \, .$$

The interchange of the integral and the duality pairing is justified since $C' \in \mathfrak{L}(Y,W')$ implies that

$$\int_0^T S'(T-s)C'y(s)\, ds \in W' \qquad \text{for } y \in L_2(0,T;Y) \, .$$

Thus, from (H3)

$$\left|\langle \int_0^T S'(T-s)C'y(s)\, ds, x\rangle_{W',W}\right| \leq c\|y\|_{L_2(0,T;Y)} \, \|ix\|_V \, .$$

(H3)' now follows from Remark 1.3.1 (v) in [24]. Q.E.D.

Let $B_\lambda = i^{-1}J_\lambda^V B$ where $J_\lambda^V = \lambda(\lambda I - A_V)^{-1}$, $\lambda \in \rho(A_V)$ on V. Note that $B_\lambda \in \mathfrak{L}(U,W)$ since $\mathrm{Range}(J_\lambda^V) = D_V(A) \subset \mathrm{Range}(i)$ by Remark. Thus, for $\lambda \in \rho(A_V)$

$$\int_0^T S(T-s)B_\lambda u(s)\, ds \in W$$

is well defined on $L_2(0,T;U)$.

Theorem 2.2. *For every* $u \in L_2(0,T;U)$ *and* $\lambda \geq \lambda_0$

$$\int_0^T S(T-s)\,B_\lambda u(s)\,ds = J_\lambda^W\, i^{-1} \int_0^T S(T-s)\,Bu(s)\,ds \ .$$

Proof. By the definition of B_λ :

$$\begin{aligned}\int_0^T S(T-s)\,B_\lambda u(s)\,ds &= \int_0^T S_W(T-s)\, i^{-1} J_\lambda^V B u(s)\,ds \\ &= \int_0^T i^{-1} S_V(T-s)\, J_\lambda^V B u(s)\,ds \\ &= i^{-1} J_\lambda^V \int_0^T S_V(T-s)\,Bu(s)\,ds \ .\end{aligned}$$

A calculation shows that for $z \in W$

$$i(\lambda I - A_W)^{-1} z = (\lambda I - A_V)^{-1}\, iz \ ,$$

thus from (H2)

$$\begin{aligned}\int_0^T S_W(T-s)\,B_\lambda u(s)\,ds &= \lambda(\lambda I - A_W)^{-1} i^{-1} \int_0^T S_V(T-s)\,Bu(s)\,ds \\ &= J_\lambda^W\, i^{-1} \int_0^T S_V(T-s)\,Bu(s)\,ds \ .\end{aligned}$$

Q.E.D.

Corollary 2.3. *For each* $\lambda \geq \lambda_0$ *define the bounded mapping* $\mathfrak{L}_\lambda$ *from* $L_2(0,T;U)$ *into* $L_2(0,T;Y)$ *by*

$$(\mathfrak{L}_\lambda u)(t) = C \int_0^T S(t-s)\, B_\lambda u(s)\, ds \ .$$

Then $\mathfrak{L}_\lambda$ converges strongly as $\lambda \longrightarrow \infty$ to $\mathfrak{L}$ where $\mathfrak{L} \in \mathfrak{L}(L_2(0,T;Y)\ ,\ L_2(0,T;Y))$ is defined by

$$(\mathfrak{L}u)(t) = C i^{-1} \int_0^T S(T-s)\, B u(s)\, ds \ .$$

Proof: Since J_λ^W converges strongly to the identity as $\lambda \longrightarrow \infty$ in W, Theorem 2.2 implies that

$$(\mathfrak{L}_\lambda u)(t) \longrightarrow (\mathfrak{L}u)(t) \quad \text{strongly, for each } t \in [0,T] \ .$$

In addition

$$\|(\mathfrak{L}_\lambda u)(t)\|_Y = \|C\|_{\mathfrak{L}(W,Y)} \, \|J_\lambda^W\|_{\mathfrak{L}(w)} \, b \|u\|_{L_2(0,T;U)} \ .$$

Thus, the corollary follows from the dominated convergence theorem. Q.E.D.

Corollary 2.4. $\mathfrak{L}_\lambda^*$ *converges strongly to* $\mathfrak{L}^*$ *as* $\lambda \longrightarrow \infty$.

Proof: It can be shown that

$$(\mathfrak{L}^* y)(t) = B'(i')^{-1} \int_t^T S'(s-t)\, C' y(s)\, ds$$

and

$$(\mathfrak{L}_\lambda^* y)(t) = B'(J_\lambda^V)'(i')^{-1} \int_t^T S'(s-t)\, C' y(s)\, ds \ .$$

The result follows from Theorem 3.1 and arguments similar to those in the proof of Corollary 2.3.

3. Main Results

Consider the optimal control problem: minimize the quadratic cost functional

$$\int_{t_0}^{T} \left(|Cx(t)|^2 + |u(t)|^2\right) dt \tag{3.1}$$

subject to

$$ix(t) = S(t - t_0)ix + \int_{t_0}^{t} S(t - s)Bu(s)ds \ .$$

Note that by using (H1), (H3) and the density of $i(W)$ in V, one can show that the operator $CS(\cdot - t_0)$ mapping W into $L_2(t_0,T;Y)$ has a unique continuous extension to all of V, and it will be denoted by $\mathcal{M}$. That is

$$\mathcal{M}x = CS(\cdot - t_0)x \quad \text{for } x \in W \tag{3.2}$$

and $\mathcal{M} \in \mathfrak{L}(V,L_2,(t_0,T;Y))$. Now the problem (3.1) can be equivalently stated as follows

$$\text{Minimize } J(u;[t_0,T]) = \left\|\mathcal{M}x + \mathfrak{L}u\right\|^2_{L_2(t_0,T;Y)} + \left\|u\right\|^2_{L_2(t_0,T;U)} \tag{3.3}$$

over $u \in L_2(t_0,T;U)$. The unique solution u^0 to (3.3) is given by

$$u^0 = -(I + \mathfrak{L}^*\mathfrak{L})^{-1}\mathfrak{L}^*\mathcal{M}x \tag{3.4}$$

and

$$\min J(u) = J(u^0) = \langle (I + \mathfrak{L}\mathfrak{L}^*)^{-1}Mx, Mx\rangle .$$

Consider the λth approximate problem of (3.3):

$$\text{(3.5)} \qquad \text{minimize } J_\lambda(u) = \|Mx + \mathfrak{L}_\lambda u\|^2 + \|u\|^2$$

over $u \in L_2(t_0,T;U)$. This problem is well posed as a class of problems discussed in [13] for $x = iz$, $z \in W$. It means that $z(t)$ is the mild solution to the evolution equation in W

$$\frac{d}{dt} z(t) = \mathcal{A}z(t) + \mathcal{B}_\lambda u(t) , \qquad z(t_0) = z \in W$$

where $\mathcal{B}_\lambda \in \mathfrak{L}(U,W)$ and $\mathcal{C} \in \mathfrak{L}(W,Y)$, and $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on W . Hence from Theorem 3.1 in [13] if $\Pi_\lambda(t)$, $t \leq T$ is the unique self-adjoint, non-negative definite solution of the Riccati equation:

$$\frac{d}{dt} \langle \Pi_\lambda(t)z,z\rangle_W + 2\langle \mathcal{A}z,\Pi_\lambda(t)z\rangle_W$$
$$= \langle \mathcal{B}_\lambda^*\Pi_\lambda(t)z,\mathcal{B}_\lambda^*\Pi_\lambda(t)z\rangle_U + \langle \mathcal{C}z,\mathcal{C}z\rangle = 0$$

for all $z \in D_W(\mathcal{A})$ and $\Pi_\lambda(T) = 0$, then the optimal solution u_λ to (3.5) (where $x = iz$) is given by

(3.7) $$u_\lambda = -B_\lambda^* \Pi_\lambda(t) U_\lambda(t,t_0) x .$$

For all $z \in W$ and $t \in [t_0,T]$

(3.8) $$\langle \Pi_\lambda(t)z, z\rangle_W = \int_t^T \left| C U_\lambda(T,s) z \right|_Y^2 ds$$

where $U_\lambda(\cdot,\cdot)$ is the perturbed evolution operator of the semigroup $S(t)$ on W by $-B_\lambda B_\lambda^* \Pi_\lambda(t)$, which means that

(3.9) $$U_\lambda(t,s) z = S(t-s) z - \int_s^t S(t-\sigma) B_\lambda B_\lambda^* \Pi_\lambda(\sigma) U(\sigma,s) z\, d\sigma$$

for $z \in W$ and $0 \leq s \leq t \leq T$. Note that (e.g., see [5], [9] and by definition of M and $\mathfrak{L}_\lambda$) for $z \in W$ and $t_0 \leq T$

(3.10) $$\Pi_\lambda(t_0) = \int_{t_0}^T S^*(s-t_0) C^*(Mz + \mathfrak{L}_\lambda u_\lambda)(s)\, ds .$$

On the other hand, problem (3.5) is also well posed for $x \in V$, and the optimal solution u_λ is given by

(3.11) $$u_\lambda = -(I + \mathfrak{L}_\lambda^* \mathfrak{L}_\lambda)^{-1} \mathfrak{L}_\lambda^* M z .$$

If j denotes the canonical isometry from W onto W', then for $z \in W$, (3.10) becomes

(3.12) $$j\Pi_\lambda(t_0)z = \int_{t_0}^{T} S'(s-t_0)\,C'(I + \mathfrak{L}_\lambda\mathfrak{L}_\lambda^*)^{-1}\,Mz(s)\,ds$$

where

$$\begin{aligned} Mz + \mathfrak{L}_\lambda u_\lambda &= Mz - \mathfrak{L}_\lambda(I + \mathfrak{L}_\lambda^*\mathfrak{L}_\lambda)^{-1}\mathfrak{L}_\lambda^* Mz \\ &= Mz - (I + \mathfrak{L}_\lambda\mathfrak{L}_\lambda^*)^{-1}\,\mathfrak{L}_\lambda\mathfrak{L}_\lambda^* Mz \\ &= (I + \mathfrak{L}_\lambda\mathfrak{L}_\lambda^*)\,Mz \end{aligned}$$

we have used

$$jS^*(\cdot)z = S'(\cdot)jz\ , \qquad z \in W$$

and

$$jC^*y = C'y\ , \qquad y \in Y\ .$$

Moreover,

(3.13) $$\min J_\lambda(u,iz) = \langle \Pi_\lambda(t_0)z,z\rangle_W = \langle j\Pi_\lambda(t_0)z,z\rangle_{W',W}$$

and $0 \leq \min J_\lambda(u,iz) \leq \beta\|iz\|_V^2$ for some positive constant β (independent of λ and t_0). From Theorem 2.1, $j\Pi_\lambda(t_0)z \in i'(V')$, $z \in W$. It then follows from the definition of M that there exists an operator $\hat{\Pi}_\lambda(t_0)$ in $\mathfrak{L}(V,V')$ such that

$$(3.14) \qquad j\Pi_\lambda(t_0)z = i'\hat{\Pi}_\lambda(t_0)iz\ , \qquad z \in W$$

From (3.12) and (3.13)

$$(3.15) \qquad \langle \Pi_\lambda(t_0)z,z\rangle_W = \langle \Pi_\lambda(t_0)iz,iz\rangle \leq \beta\|iz\|^2$$

for $z \in W$. Since $\Pi_\lambda(t_0)$ is self-adjoint on W and $(V')' = V$;

$$\|\hat{\Pi}_\lambda(t_0)\|_{\mathfrak{L}(V,V')} \leq \beta \quad \text{and} \quad \hat{\Pi}_\lambda(t_0) \text{ is summetric}$$

in the sense that $\hat{\Pi}_\lambda(t_0)' = \hat{\Pi}_\lambda(t_0)$.

We now have the following lemma.

Lemma 3.1. *If* u^0 *and* u_λ *are defined by* (3.3) *and* (3.7) *respectively, then* u_λ *converges strongly to* u^0 *as* $\lambda \longrightarrow \infty$ *in* $L_2(t_0,T;U)$ *for all* $x \in V$, *and the convergence is uniform in* $t_0 \in [0,T]$.

Proof: Since

$$(I + \mathfrak{L}_\lambda^*\mathfrak{L}_\lambda)^{-1} - (I + \mathfrak{L}^*\mathfrak{L})^{-1}$$
$$= (I + \mathfrak{L}_\lambda^*\mathfrak{L}_\lambda)^{-1}\ (\mathfrak{L}_\lambda^*\mathfrak{L}_\lambda - \mathfrak{L}^*\mathfrak{L})\ \ (I + \mathfrak{L}^*\mathfrak{L})^{-1}$$

and $\|(I + \mathfrak{L}_\lambda^*\mathfrak{L}_\lambda)^{-1}\| \leq 1$ uniformly in λ , it follows from Corollaries 2.3 and 2.4 that

$$(I + \mathfrak{L}_\lambda^* \mathfrak{L}_\lambda)^{-1} \longrightarrow (I + \mathfrak{L}^* \mathfrak{L})^{-1} \quad \text{strongly.}$$

The lemma results from (3.3) and (3.7). Q.E.D.

Define the evolution operator $\mathcal{U}(t,t_0)$, $0 \leq t_0 \leq t \leq T$ on V by

$$U(t,t_0)x = S(t-t_0)x + \int_{t_0}^{t} S(t-s)\mathcal{B}u_{t_0}^0(s)\,ds \,, \tag{3.16}$$

where $u_{t_0}^0$ is the optimal solution to (3.3) in the interval $[t_0,T]$. Then the following theorem holds.

Theorem 3.2.

(i) $\quad \mathcal{U}(t,t) = I \,, \quad t \in [0,T]$.

(ii) $\quad \mathcal{U}(t,s)\,\mathcal{U}(s,t_0) = \mathcal{U}(t,t_0)$ *for* $0 \leq t_0 \leq s \leq t \leq T$.

(iii) $\quad \mathcal{U}(t,t_0)$ *is jointly continuous in* t *and* t_0 *on* V, H, *and* W, *respectively..*

(iv) *The operator* $z \in W \longrightarrow C\mathcal{U}(T,\cdot)z \in L_2(0,T;Y)$ *has a continuous extension to all of* $x \in V$.

Proof: Property (ii) follows from the principle of optimality; i.e., if u^0 is the optimal solution to (3.3) on the interval $[t_0,T]$, then for $t_0 \leq s \leq T$, $u^0\chi_{[s,T]}$ is the optimal solution to (3.3) on the interval $[s,T]$ with initial condition $x^0(s) = \mathcal{U}(s,t_0)x$.

Note that for $z \in W$

$$iU(t,t_0)z = S(t-t_0)iz + \int_0^t S(t-t_0)Bu_{t_0}(s)\,ds \ .$$

For property (iii), from (H1) it suffices to show that for $x \in V$

$$i^{-1}\int_{t_0}^t S(t-s)Bu_{t_0}^0(s)\,ds$$

is jointly continuous on W. The continuity with respect to t_0 follows from (H3) and the fact that $u_t^0\,\chi_{[t_0,T]}(\cdot)$ is strongly continuous in $L_2(0,T;U)$. In order to show the continuity in t, first let $\Delta t \geqslant 0$. Then

$$\int_{t_0}^{t+\Delta t} S(t+\Delta t-s)Bu_{t_0}^0(s)\,ds - \int_{t_0}^t S(t-s)Bu_{t_0}^0(s)\,ds$$
$$= (S(\Delta t)-I)\int_{t_0}^t S(t-s)Bu_{t_0}^0(s)\,ds + \int_t^{t+\Delta t} S(t+\Delta t-s)Bu_{t_0}^0(s)\,ds$$

and we then obtain

$$\text{(3.17)} \qquad \left\| i^{-1}\left[\int_{t_0}^{t+\Delta t}\cdot - \int_{t_0}^t\cdot\right]\right\|_W$$
$$\leqslant \left\| (S(\Delta t)-I)i^{-1}\int_{t_0}^t S(t-s)Bu_{t_0}^0(s)\,ds\right\|_W + b\left\|u_t^0\right\|_{L_2(t,t+\Delta t;U)} \ .$$

The first term on the right-hand side of (3.17) goes to zero by the strong continuity of $S(t)$ on W, and the convergence to zero of the second term is a standard analysis result. The proof for $\Delta t \leqslant 0$ is similar.

Property (iv) follows from the above result and (3.2). Q.E.D.

Now we can state the extended result of Theorem 3.1 in [13].

Theorem 3.3. $\hat{\Pi}_\lambda(t_0)$ *converges strongly to a symmetric operator* $\Pi(t_0)$ *in* $\mathfrak{L}(V,V')$ *and the convergence is uniform in* $t_0 \in [0,T]$. *Moreover, for* $x \in V$

$$\min J(u,x) = \langle \Pi(t_0)x, x\rangle_{V',V}$$

$$= \int_{t_0}^{T} \left\| \mathcal{C}U(T,s)x \right\|_Y^2 \, ds .$$

Proof: It follows from (3.12) and (3.14) that

$$i'\hat{\Pi}_\lambda(t_0)x = \int_{t_0}^{T} S'(s-t_0)\,\mathcal{C}'(Mx + \mathfrak{L}_\lambda u_\lambda)(s)\,ds .$$

Thus, from Theorem 2.1, Corollary 2.3, and Lemma 3.1 we have

$$\lim_{\lambda \uparrow \infty} i'\hat{\Pi}_\lambda(t_0) = i'\Pi(t_0)x$$

$$= \int_{t_0}^{T} S'(s-t_0)\,\mathcal{C}'(Mx + \mathfrak{L}u^0)(s)\,ds$$

and the convergence is uniform in $t_0 \in [0,T]$. From (3.9) and Theorem 2.2, we have that for $z \in W$

$$\mathcal{U}_\lambda(T,t_0)z = S(T-t_0)z + J_\lambda^W i^{-1} \int_{t_0}^{T} S(T-s)\mathcal{B}u_\lambda(s)\,ds .$$

It then follows from Lemma 3.1 and the fact that $J_\lambda^W z \longrightarrow z$ (strongly) in W as $\lambda \longrightarrow \infty$, that for each $t_0 \leq T$

(3.18) $$\mathcal{U}_\lambda(T,t_0)z \longrightarrow \mathcal{U}(T,t_0)z \quad \text{strongly in } W .$$

Since $\|J_\lambda^W\|$ and $\|u_\lambda\|_{L_2(t_0,T;U)}$ are uniformly bounded in λ and $t_0 \in [0,T]$ the dominated convergence theorem implies that

$$C\mathcal{U}_\lambda(T,\cdot)z \longrightarrow C\mathcal{U}(T,\cdot)z \quad \text{in} \quad L_2(0,T;Y) .$$

Thus, from (3.8), (3.15) and the convergence of $\hat{\Pi}_\lambda(t_0)$ to $\Pi(t_0)$ we obtain that for $z \in W$ and $t_0 \leq T$

$$\langle \Pi(t_0)iz, iz\rangle_{V',V} = \int_{t_0}^{T} \left\| C\mathcal{U}(T,s)z \right\|_Y^2 ds .$$

The desired result now follows from (iv) of Theorem 3.2 and the density of $i(W)$ in V. Q.E.D.

Theorem 3.4. $\Pi(t) \in C_s(0,T;\mathfrak{L}(V,V'))$.

Proof: *For the moment, let us indicate the dependence on* t_0 *of the operator* $\mathcal{M}$ and $\mathfrak{L}$ introduced for the optimal control problem (3.3) and write $\mathcal{M}_{t_0}$ and

$\mathfrak{L}_{t_0}$, respectively. It is easily verified that M_{t_0}, $\mathfrak{L}_{t_0}$, and $\mathfrak{L}^*_{t_0}$ are strongly continuous in t_0 on $[0,T]$. Recall that for $x \in V$

$$\langle \Pi(t_0)x, x \rangle_{V',V} = \min J(u;[t_0,T])$$

$$= \langle (I + \mathfrak{L}_{t_0} \mathfrak{L}^*_{t_0})^{-1} M_{t_0}x, M_{t_0}x \rangle .$$

Using arguments similar to those in the proof of Lemma 3.1, it can be shown that $(I + \mathfrak{L}_{t_0} \mathfrak{L}^*_{t_0})^{-1}$ is strongly continuous in t_0, thus it follows that $\langle \Pi(t_0)x,x \rangle_{V',V}$ is a non-increasing continuous function in t_0 on $[0,T]$. If j_V denotes the canonical isometry from V' onto V, then for $x,y \in V$

$$\langle j_V\Pi(t_0)x, y \rangle_V = \langle \Pi(t_0)x, y \rangle_{V',V}$$

$$= \langle x, \Pi(t_0)y \rangle_{V,V'}$$

$$= \langle x, j_V\Pi(t_0)y \rangle_V$$

where we used the symmetry of $\Pi(t_0)$. Thus, $j_V\Pi(t_0)$ is self-adjoint on V. It now follows from [16, p. 454, Theorem 3.3] that $j_V\Pi(t_0)$ is strongly continuous in V for $x \in V$. The result follows since j_V is isometric.

Q.E.D.

Corollary 3.5. *The optimal solution* u^0 *is given by*

(3.19) $$u^0(t) = -B'\Pi(t)\,U(T,t_0)\,x$$

where $U(\cdot,\cdot)$ *is the evolution operator on* V *defined by* (3.16) *satisfies*

$$(3.20) \qquad iU(t,s)z = S(t-s)iz - \int_s^t S(t-\sigma)BB'\Pi(\sigma)iU(\sigma,s)z\,d\sigma$$

for $z \in W$ and $0 \leq s \leq t \leq T$.

Proof: For $z \in W$ and $u \in U$

$$\langle B_\lambda^* z, u\rangle_U = \langle z, B_\lambda u\rangle_W$$

$$= \langle jz, i^{-1}J_\lambda^V Bu\rangle_{W',W} .$$

If

$$jz \in i'(V') = \langle B'(J_\lambda^V)'(i')^{-1}\, jz, u\rangle_U ,$$

thus $B_\lambda^* z = B_\lambda^*(J_\lambda^V)'(i')^{-1}jz$ for $jz \in i'(V')$. Note that (3.14) shows that $j\Pi_\lambda(t) \in i'(V')$ and that

$$B_\lambda^*\Pi_\lambda(t)\,U_\lambda(t,t_0)z = B'(J_\lambda^V)'\,\hat{\Pi}_\lambda(t)\,iU_\lambda(t,t_0)z$$

for $z \in W$. By Theorem 3.3, (3.18) and the fact that $(J_\lambda^V) \longrightarrow I$ on V', we obtain

$$B_\lambda^*\Pi_\lambda(t)\,iU_\lambda(t,t_0)z \longrightarrow B'\Pi(t)\,iU(t,t_0)z , \qquad z \in W .$$

It then follows from Lemma 3.1 that

(3.21) $$u^0(t) = \lim_{\lambda\uparrow\infty} u_\lambda(t) = -B'\Pi(t)\, i\,U(t,t_0)\,z\ , \qquad z \in W\ .$$

Since (3.4) and the right-hand side of (3.21) depend continuously on $x \in V$, (3.19) holds for all $x \in V$ and hence (3.20) follows from Theorem 3.2.

Q.E.D.

The form of the optimal control is often written as

(3.22) $$u^0(t) = -K(t)U(t,t_0)\,x$$

where the operator $K(t) = B'\Pi(t) \in C_s(0,T;\mathfrak{L}(V,U))$ is called the optimal gain operator. Recall that the operator $CS(\cdot - t_0) : W \longrightarrow L_2(t_0,T;Y)$ has a continuous extension M_{t_0} on V (see, (3.2)). Thus, for each $u \in U$

$$M_{t_0} Bu \in L_2(t_0,T;Y)$$

and if $\dim(U)$ is finite, this implies that

$$\left\| M_{t_0}(\cdot)B \right\|_{\mathfrak{L}(U,Y)} \quad \text{is square integrable}$$

on $[t_0,T]$. Define $L(t)$ as the unique bounded extension of $CU(T,t) : W \longrightarrow L_2(0,T;Y)$ on V (see Theorem 3.2 (iv)). Then we have the following result.

Theorem 3.6. *Assume* $\dim(U)$ *is finite and let* Z *be as in (H4). Then* $K(t)x$, $x \in V$ *and* $L(t)z$, $z \in Z$ *are absolutely continuous on* $[0,T]$ *in* U *and* Y *respectively. Moreover,* $K(t)$ *and* $L(t)$ *satisfy the Chandrasekhar equations:*

$$\text{(3.23)}\qquad \begin{aligned} &\frac{d}{dt} K(t)x = -B'L'(t)L(t)x , \qquad x \in V \\ &K(T) = 0 \end{aligned}$$

and

$$\text{(3.24)}\qquad \begin{aligned} &\frac{d}{dt} L(t)z = -L(t)(A - BK(t))z , \qquad z \in Z \\ &L(T) = C . \end{aligned}$$

Proof: From (3.20) we have

$$L(t)B = M_{t_0}(t)B - Ci^{-1}\int_t^T S(T-s)BK(s)U(s,t)B\,ds .$$

Thus, from (H2) $\|L(t)B\|_{\mathfrak{L}(U,Y)}$ is square integrable on $[0,T]$ and so is $\|(L(t)B)^*\| = \|B'L'(t)\|$. By Theorem 3.3, for $x \in V$ and $u \in U$

$$\begin{aligned} \langle K(t)x,u\rangle_U &= \langle B'\Pi(t)x,u\rangle \\ &= \langle \Pi(t)x,Bu\rangle_{V',V} \\ &= \int_t^T \langle L(s)x,L(s)Bu\rangle_Y \\ &= \langle \int_t^T (L(s)B)^* L(s)x\,ds,u\rangle_U . \end{aligned}$$

This implies that

$$(3.25) \qquad K(t)x = \int_t^T B'L'(s)L(s)x\,ds$$

where the integrand is U -valued integrable. The differential equation (3.23) for $K(t)$ now follows immediately.

Note that for $z \in Z$, $t \longrightarrow \mathcal{U}(T,t)z$ is continuously differentiable in V and

$$(3.26) \qquad \mathcal{U}(T,t)z - z = \int_t^T \mathcal{U}(T,s)\,(\mathcal{A} - \mathcal{B}K(s))z\,ds\ .$$

If $z \in \mathcal{D}_V(\mathcal{A}^2)$, then $\mathcal{A}z \in Z \subset W$ and from (3.26) and the fact that $\|L(t)B\|$ is square integrable,

$$\mathcal{C}\mathcal{U}(T,t)z - \mathcal{C}z = \int_t^T \mathcal{C}\mathcal{U}(T,s)\,(\mathcal{A} - \mathcal{B}K(s))z\,ds\ .$$

Since $L(t)$ is the bounded extension of $\mathcal{C}\mathcal{U}(T,\cdot) : W \longrightarrow L_2(0,T;Y)$ and $\mathcal{D}_V(\mathcal{A}^2)$ is dense in Z , $L(t)$ satisfied

$$L(t)z = \mathcal{C}z + \int_t^T L(s)\,(\mathcal{A} - \mathcal{B}K(s))z\,ds\ , \qquad z \in Z$$

and the theorem follows. Q.E.D.

The following theorem shows the uniqueness of solutions of (3.23) and (3.24).

Theorem 3.7. *Assume* $\dim(U)$ *is finite. The equation (3.23)-(3.24) has a unique solution within a class of operators such that*

$$K(\cdot) \in C_s(0,T;\mathfrak{L}(V,U))$$

and

$$L(\cdot) \in C_s(0,T;\mathfrak{L}(W,Y)) \cap \{L(\cdot)x \in L_2(0,T;Y) \text{ for all } x \in V\} .$$

Proof: Suppose (K, L) and $(\hat{K},\hat{L})$ are solutions to (3.23)-(3.24). Then for $z \in Z$

$$\begin{aligned}\frac{d}{dt}(L(t)-\hat{L}(t))z &= -L(t)(A-BK(t))z + \hat{L}(t)(A-B\hat{K}(t))z \\ &= -(L-\hat{L})(A-BK(t))z + \hat{L}(t)B(K-\hat{K})z .\end{aligned}$$

Since $\dim(U)$ is finite, $\|\hat{L}(\cdot)B\|_{\mathfrak{L}(U,Y)}$ is square integrable. Let us denote by $U(t,s)$ the evolution operator on V generated by $A - BK(\cdot)$. Then, for $x \in V$

$$L(t)x - \hat{L}(t)x = \int_t^T \hat{L}(s)B(K(s)-\hat{K}(s))\,U(s,t)x\,ds . \tag{3.27}$$

From (3.23), for $x \in V$,

$$\begin{aligned}&\langle K(t)x - \hat{L}(t)x, u\rangle_U \\ &\quad = \int_t^T \langle (L(s)B - \hat{L}(s)B)u, L(s)x\rangle_Y\,ds + \int_t^T \langle \hat{L}(s)Bu, L(s) - \hat{L}(s)x\rangle ds .\end{aligned}$$

From (3.27), $L - \hat{L} \in C_s(0,T;\mathfrak{L}(V,Y))$ and thus this implies that for $x \in V$

$$\left\|K(t)x - \hat{K}(t)x\right\|_U \leq \int_t^T \left\|L(s)x\right\|_Y \left\|L(s)B - \hat{L}(s)B\right\|_{\mathfrak{L}(U,Y)} ds$$

$$+ \int_t^T \left\|L(s)x - \hat{L}(s)x\right\|_Y \left\|\hat{L}(s)B\right\|_{\mathfrak{L}(U,Y)} ds ,$$

or equivalently,

$$\left\|K(t)x - \hat{K}(t)x\right\|_U^2 \leq 2\int_0^T \left\|L(s)x\right\|_Y^2 ds \int_0^T \left\|L(s)B - \hat{L}(s)B\right\|_{\mathfrak{L}(U,Y)}^2 ds \tag{3.28}$$

$$+ 2\int_0^T \left\|\hat{L}(s)B\right\|_{\mathfrak{L}(U,Y)}^2 \int_t^T \left\|L(s)x - \hat{L}(s)x\right\|_Y^2 ds .$$

Similarly, (3.27) yields that

$$\left\|L(t) - \hat{L}(t)\right\|_{\mathfrak{L}(V,Y)}^2 \leq M_1^2 M_2 \int_t^T \left\|K(s) - \hat{K}(s)\right\|_{\mathfrak{L}(V,U)}^2 ds$$

where

$$M_1 = \max_{0 \leq s \leq t \leq T} \left\|U(s,t)\right\|_{\mathfrak{L}(V)} \quad \text{and} \quad M_2 = \int_0^T \left\|\hat{L}(s)B\right\|^2 ds .$$

Thus, (3.28) implies that

$$\left\|K(t) - \hat{K}(t)\right\|_{\mathfrak{L}(V,U)}^2 \leq \left[2M_3 \left\|B\right\|_{\mathfrak{L}(U,V)}^2 + 2M_2\right] \int_t^T \left\|L(s) - \hat{L}(s)\right\|_{\mathfrak{L}(V,Y)}^2 ds$$

where

$$\int_0^T \|L(s)x\|_Y^2 \, ds \leqslant M_3 \|x\|_Y^2 .$$

Hence, the result follows from Gronwall's lemma. Q.E.D.

By [10, p. 109, Corollary 2.10] we have that if $t \longrightarrow f(t) \in V$ is absolutely continuous on $[0,T]$, then the function

$$v(t) = \int_0^T S(t-s)f(s)\,ds \in D_V(A) , \qquad t \geqslant 0$$

satisfies the differential equation

$$\frac{d}{dt} v(t) = Av(t) + f(t) \qquad \text{a.e.}$$

Thus using a similar argument to those in the proof of Lemma 4.2 in [13], one can show

Theorem 3.8. *Assume* $\dim(U)$ *is finite. Then, the evolution operator defined by* (3.16) *and* (3.20) *has the following properties: for* $z \in Z$ *and* $0 \leqslant s \leqslant t \leqslant T$, $t \longrightarrow U(t,s)z \in V$ *is continuously differentiable,* $U(t,s)z \in Z$ *and*

$$\frac{\partial}{\partial t} U(t,s)z = (A - BK(t))U(t,s)z .$$

Corollary 3.9. *For any* $x \in Z$, *the optimal solution* u^0 *to* (3.1) *is absolutely continuous on* $[0,T]$.

Proof. From Theorem 3.8, for $x \in Z$, $\mathcal{U}(t,t_0)x \in Z \subset W$ and $t \longrightarrow \mathcal{U}(t,t_0)x \in V$ is continuously differentiable. Thus from (3.22) and (3.25)

$$\frac{d}{dt}\, u^0(t) = -K(t)(\mathcal{A} - \mathcal{B}K(t))\,\mathcal{U}(t,t_0)x + \mathcal{B}'L'(t)L(t)\,\mathcal{U}(t,t_0)\,x$$

where we have used $L(\cdot) \in C_s(0,T;\mathfrak{L}(W,Y))$. Q.E.D.

4. Examples

As shown in [22], the general framework in Section 2 applies to a wide class of problems; e.g., the neutral functional differential equation (FDE) with delays in quadratic cost [14], the parabolic partial differential equation (PDE) with Neumann or mixed type boundary control, and the retarded FDE with delays in control and quadratic cost. Thus, the results in Section 3 apply to these problems.

The other example which can be discussed within the framework of Section 2 is the following: consider a retarded FDE in $\mathbb{R}^n$ with delays in control [6], [12], [27]

$$
(4.1)\qquad \begin{aligned}
&\dot{x}(t) = \int_{-r}^{T} d\mu(\theta)\, x(t+\theta) + \int_{-r}^{0} dB(\theta)\, u(t+\theta) \\
&x(0) = \eta, \quad x(\theta) = \phi(\theta) \quad \text{and} \quad u(\theta) = v(\theta), \quad -r \leqslant \theta < 0 ,
\end{aligned}
$$

where $\mu(\cdot)$ and $B(\cdot)$ are $n \times n$ and $n \times m$ matrix valued functions of bounded variation which vanish at $\theta = 0$ and are left continuous on $(-r,0)$. Let us consider the linear quadratic optimal control problem; for given $((\eta,\phi),v) \in \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^m)$ choose the control $u \in L_2(0,T;\mathbb{R}^m)$ that minimizes the cost functional

$$
(4.2)\qquad J(u;[0,T]) = \int_0^T (|Cx(t)|^2 + |u(t)|^2)\, dt
$$

where C is a $p \times n$ matrix with $p \leqslant n$.

Define a structure operator $\mathcal{F}$ on $\mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n) \times L_2(-r,0;\mathbb{R}^n)$ by

$$\mathcal{F}(\eta,\phi,v) = \left[\eta,\ \int_{-r}^{\theta-} d\mu(\xi)\,\phi(\xi-\theta) + \int_{-r}^{\theta-} d\beta(\xi)\,v(\xi-\theta)\right]$$

$$\in \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)\ .$$

It is shown in [12], [27] that the function $z(t) = \mathcal{F}(x(t),x(t+\cdot),u(t+\cdot))$ satisfies

$$\frac{d}{dt}\,z(t) = A_T^{\prime}z(t) + B_T^{\prime}u(t) \qquad \text{in } V$$

where A_T is the infinitesimal generator of a strongly continuous semigroup on $H = \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$ defined by

$$D(A_T) = \{(\eta,\phi) \in \mathbb{R}^n \times L_2 \,|\, \dot{\phi} \in L_2 \ \text{ and } \ \eta = \phi(0)\}$$

and

$$A_T(\phi(0),\phi) = \left[\int_{-r}^{0} d\mu^T(\theta)\,\phi(\theta),\ \dot{\phi}\right] \in \mathbb{R}^n \times L_n \quad \text{for } (\phi(0),\phi) \in D(A_T)\,,$$

B_T defined on $D(A_T)$ is given by

$$B_T(\phi(0),\phi) = \int_{-r}^{0} d\beta^T(\theta)\,\phi(\theta)\ ,$$

and

$$V = D(A_T)' \subset H = H' \subset D(A_T)\,.$$

Then the cost functional (4.2) is equivalently written as

$$J(u;[0,T]) = \int_0^T (|\mathcal{C}z(t)|^2 + |u(t)|^2)\,dt$$

where $\mathcal{C}(\eta,\phi) = C\eta,(\eta,\phi) \in \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$. If we take $H = W = \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$ and $V = \mathcal{D}(\mathcal{A}_T)'$, then the conditions (H1), (H2)', (H3)', and (H4) are satisfied (see Lemma 5.1 in [13]). By duality, hypothesis (H1) ~ (H4) are satisfied and thus the results in Section 3 apply to this example; i.e., the optimal control u^0 to (4.1)-(4.2) is given by

$$u^0(t) = -K(t)\mathcal{F}(x^0(t), x^0(t+\cdot), u^0(t+\cdot))$$

where $x^0(t)$ is the optimal trajectory of (4.1) corresponding to u^0 and the optimal gain operator $K(t)$ satisfies

$$\frac{d}{dt}K(t) = -\mathcal{B}_T L'(t)L(t)x\,, \qquad x \in V$$

$$K(T) = 0$$

and

$$\frac{d}{dt}L(t) = -L(t)(\mathcal{A}_T' - \mathcal{B}_T'K(t))z\,, \qquad z \in H$$

$$L(T) = \mathcal{C}\,.$$

5. Boundary Control Problems

In this section, we discuss problems which cannot be handled by the results in Sections 2 and 3. The problems which will be disucssed can be formulated as the boundary control problem [7];

$$
\begin{aligned}
&\frac{d}{dt}\, x(t) = Ax(t)\,, \qquad x(0) = x \in H \\
(5.1)\qquad & \\
&\tau x(t) = u(t)
\end{aligned}
$$

where A is a closed operator on a Hilbert space H and τ is a linear operator from H onto the Hilbert space U and the restriction of τ to $\mathrm{dom}(A)$ is continuous with respect to the graph norm of A. Define the associated operator $\mathcal{A}$ on H by

$$D(\mathcal{A}) = \{x \in \mathrm{dom}(A) \text{ and } \tau x = 0\}$$

and

$$\mathcal{A}x = Ax \quad \text{for} \quad x \in D(\mathcal{A})\,.$$

We assume that $\mathcal{A}$ generates a strongly continuous semigroup $S(t)$ on H and moreover we assume that there exists a Green map $G : U \longrightarrow \mathrm{dom}(A)$ such that

$$AGu = 0 \quad \text{and} \quad \tau Gu = u \quad \text{for all } u \in U\,.$$

Then one can write (5.1) as the form of (2.1) and (2.2) [17];

$$x(t) = S(t)x + \int_0^t S(t-s)\mathcal{B}u(s)\,ds \quad \text{in} \quad V \tag{5.2}$$

where $\mathcal{B}u = -\mathcal{A}Gu$, $u \in U$ and $V = \mathcal{D}(A^*)'$. Since $\mathcal{A} \in \mathfrak{L}(H,V)$ [24, Lemma 1.3.2], $\mathcal{B}u \in V$, $u \in U$. We will discuss the following three cases of interests.

5.1 Evolution Equations with Delays in Control [11]

Consider the control system with delays in control:

$$(5.3)\qquad \begin{aligned} &\frac{d}{dt} z(t) = A_0 z(t) + B_0 u(t) + A_{01} u(t + \cdot) \\ &z(0) = z \in H_0 \quad \text{and} \quad u(\theta) = v(\theta)\ , \quad -r \leq \theta \leq 0 \end{aligned}$$

where A_0 is the infinitesimal generator of a strongly continuous semigroup $S_0(t)$ on H_0 and A_{01} is a linear operator on $L_2(-r,0;U)$ defined by

$$A_{01} y = \sum_{i=1}^{k} B_i y(\theta) + \int_{-r}^{0} B(\theta)\, y(\theta)\, d\theta$$

where $-r = \theta_k < \theta_{k-1} < \cdots \theta_1 < \theta_0 = 0$, $B_i \in \mathfrak{L}(U,H_0)$, and $B(\cdot) \in \mathfrak{L}(U,H)$ is strongly measurable and $\theta \longrightarrow \|B(\theta)\|_{\mathfrak{L}(U,H)}$ is integrable on $[-r,0]$. Let us consider the linear quadratic optimal control problem: for given $x \in H_0$ and $v \in L_2(-r,0;U)$ minimize the cost functional

$$(5.4)\qquad J(u,[0,T]) = \int_0^T \left(\|Cz(t)\|_Y^2 + \|u(t)\|_U^2 \right) dt$$

where $C \in \mathfrak{L}(H_0,Y)$.

Let $y(t,\theta) = u(t + \theta)$, $t \geq 0$ and $-r \leq \theta \leq 0$, then one can write (5.3) as a boundary control problem (5.1):

$$\frac{d}{dt} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & A_{01} \\ 0 & D \end{bmatrix} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(t)$$

$$y(t,0) = u(t)$$

with $H = H_0 \times L_2(-r,0;U)$ where

$$Dy = \frac{d}{d\theta} y \;, \quad u \in L_2(-r,0;U)$$

with domain

$$\mathcal{D}(D) = \{y \in L_2(-r,0;U) \mid y \text{ is absolutely continuous and } \dot{y} \in L_2\} \;.$$

It is shown in [11] that the associated generator $\mathcal{A}$ with domain $\mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(D_0)$ where $\mathcal{D}(D_0) = \{y \in \mathcal{D}(D) \mid y(0) = 0\}$, generates a strongly continuous semigroup $S(t)$ on H and that

$$-\mathcal{A}Gu = \begin{bmatrix} 0 \\ \mathcal{B}_1 u \end{bmatrix}, \quad u \in U$$

where $\mathcal{B}_1' y = y(0)$.

Thus, one can write (5.3)-(5.4) as the control problem of (3.1) with $V = D(A_0^*)' \times \mathcal{D}(D_0^*)'$, $H = W = H_0 \times L_2(-r,0;U)$, where

$$\mathcal{D}(D_0^*) = \{y \in \mathcal{D}(D) \mid y(-r) = 0\}$$

and

$$\mathcal{D}(D_0^*) \subset L_2 \subset L_2' \subset \mathcal{D}(D_0^*)' \;.$$

For this example, one can show that (H1), (H2), and (H4) hold [11]. However, (H3) is not satisfied unless A_0 generates an analytic semigroup. Instead, we have the following properties. The solution semigroup $S(t)$ on H is given by

$$S(t) = \begin{bmatrix} S_0(t) & S_{01}(t) \\ 0 & S_1(t) \end{bmatrix}$$

where for $y \in L_2(-r,0;U)$

$$(S_1(t)y)(\theta) = y(t+\theta)\chi_{[-r,-t]}(\theta), \quad -r \leq \theta \leq 0$$

and

$$S_{01}(t)y = \int_0^t S_0(t-s)A_{01}S_1(s)y\,ds \in H .$$

A calculation shows that for $u \in U$

$$S_{01}(t)\mathcal{B}_1 u = \sum_{i=1}^{k} \tilde{S}_0(t+\theta_i)B_i u + \int_{-r}^{0} \tilde{S}_0(t+\theta)B(\theta)u\,d\theta$$

where $\tilde{S}_0(\cdot)$ is defined by

$$\tilde{S}_0(t)z = \begin{cases} S_0(t)z, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad z \in H_0 ,$$

and thus

$$(5.5) \qquad CS(t)B = CS_0(t)B_0 + CS_{01}(t)B_1 \in L^\infty(0,T;\mathfrak{L}(U,H)) \cap C_s(r,T;\mathfrak{L}(U,H)) .$$

Let for $\lambda \geq 0$

$$B_\lambda = \begin{bmatrix} B_0 \\ \lambda(\lambda I - D_0)^{-1} B_1 \end{bmatrix} .$$

Then $B_\lambda \in \mathfrak{L}(U,H)$. Thus, one can apply Theorem 3.1 in [13] to the system defined by the triple (A,B_λ,C) and using Proposition 2.1, Lemmas 2.2-2.3, and Theorem 2.3 in [11], one can then obtain that for $t_0 \leq T$

$$u^0_{t_0}(t) = -B'\Pi(t)U(t,t_0)x$$

and

$$\langle \Pi(t)x,x\rangle_H = \int_t^T |CU(T,s)x|^2 ds \quad \text{for all} \quad x = (z,v) \in H$$

where the evolution operator $U(t,s)$ is jointly continuous on $0 \leq s \leq t \leq T$ and satisfies

$$(5.6) \qquad U(t,t_0)x = S(t-t_0)x + \int_{t_0}^t S(t-s)Bu^0_{t_0}(s)\,ds ,$$

and $B'(z,y) = B_0^* z + y(0)$. Let $L(t)x = CU(T,t)x$ for $x \in H$ and $t \leq T$. Recall that if for $0 \leq t_0 \leq T$, $\mathfrak{L}_{t_0}$ and M_{t_0} are in Sections 2 and 3, then the optimal control $u^0_{t_0}$ on the interval $[t_0,T]$ is given by

$$u^0_{t_0} = -(I + \mathcal{L}^*_{t_0}\mathcal{L}_{t_0})^{-1}\mathcal{L}^*_{t_0} M_{t_0} x \,, \quad x \in H \,. \tag{5.7}$$

Note that from (5.5), $t_0 \longrightarrow M_{t_0}\mathcal{B}u \in L_2(0,T;Y)$ is strongly continuous for each $u \in U$. Also from (5.5), $\mathcal{L}_{t_0}$ and $\mathcal{L}^*_{t_0}$ are strongly continuous, which means that $(I + \mathcal{L}^*_{t_0}\mathcal{L}_{t_0})^{-1}\mathcal{L}^*_{t_0}$ is strongly continuous in t_0 (see the proof of Lemma 3.1). It now follows from (5.5), (5.6), and (5.7) that $L(t)\mathcal{B}$ is piecewise continuous in norm on $[0,T]$. Moreover, one can show that $L(t)x$, $t \leq T$ satisfied

$$L(t)x = \mathcal{C}S(T-t)\,x - \int_t^T L(s)\mathcal{B}\mathcal{B}'\Pi(s)\,S(s-t)\,x\,ds \,, \qquad x \in H$$

(see Lemma 5.4 for its derivation).

Using arguments similar to those in the proof of Theorem 3.6, we obtain the optimal feedback gain operator $K(t) = \mathcal{B}'\Pi(t)$, $t \leq T$ is given by

$$K(t)\,x = \int_t^T (L(s)\mathcal{B})^*\,L(s)\,x\,ds$$

and thus $t \longrightarrow K(t)x$, $x \in H$ and $t \longrightarrow L(t)x$, $x \in \mathcal{D}(A)$ is piecewise continuously differentiable on $[0,T]$. As in Section 3, $K(t)$ and $L(t)$ satisfy the equations (3.23) and (3.24).

5.2 Hyperbolic Systems [18], [23]

Consider the second-order hyperbolic system with Dirichlet boundary control:

$$
(5.8)\qquad \begin{cases} \dfrac{\partial}{\partial t^2}\, y(t,\xi) + A_0 y(t,\xi) = 0\ , & \xi \in \Omega \\[2ex] y(0,\cdot) = y_0 \ \text{ and } \ \dfrac{\partial}{\partial t} y(0,\cdot) = y\ , & \\[2ex] y(t,\sigma) = u(t,\sigma)\ , & \sigma \in \Gamma \end{cases}
$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and A_0 be a second-order uniformly strong elliptic operator in Ω. One can formulate (5.8) as the evolution of (2.1):

$$
\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}_0 G \end{bmatrix} u(t)
$$

where $x_1(t) = (y(t,\cdot)$ and $x_2(t) = (\partial/\partial t)y(t,\cdot)$ and $u(t) = u(t,\cdot)$, G is the Green map which satisfies

$$
(5.9)\qquad Gu|_\Gamma = u \quad \text{and} \quad A_0 Gu = 0 \quad \text{in} \quad \Omega\ ,
$$

and $\mathcal{A}_0$ is defined by $\mathcal{D}(\mathcal{A}_0) = H_0^1(\Omega) \cap H^2(\Omega)$ and $\mathcal{A}_0 x = A_0 x$, $x \in \mathcal{D}(\mathcal{A}_0)$. Here note that $\mathcal{A}_0 G \in \mathcal{D}(\mathcal{A}_0)'$. Let $H = W = L^2(\Omega) \times H_0^1(\Omega)$ and $V = H_0^1(\Omega)' \times \mathcal{D}(\mathcal{A}_0)'$ where $L^2(\Omega)$ is taken as the pivoting space. If $\mathcal{A}$ is the associated generator on H with domain $\mathcal{D}(\mathcal{A}) = H_0^1(\Omega) \times L^2(\Omega)$, then $V = \mathcal{D}(\mathcal{A}^*)'$ and by Hille-Yosida theorem $\mathcal{A}$ generates a strongly continuous semigroup both on H and V , and thus hypothesis (H1) holds. Under appropriate conditions, it is shown in [18] that (H2) holds. However, (H3) is not satisfied in general unless $\mathrm{Range}(C^*) \subset D(A)$.

Motivated by this example, we consider the case when instead of (H3), the condition

(H5) $\dim(Y)$ is finite

is assumed, $H = W$, $V = D(A^*)'$, and (H2) holds. Under (H5), we shall first show that Corollary 2.4 holds. Recall the statement (H2)' of Theorem 2.1.

$$(5.10) \qquad \|B'S'(T-\cdot)x\|_{L_2(0,T;U)} \leq b\|x\|_H^2 \quad \text{for } x \in V' .$$

Let us denote by $\overline{B'S'(T-\cdot)}$, the bounded extension of $x \in V' \longrightarrow B'S'(Y-\cdot)x \in L_2(0,T;U)$ on H . Since $C^* \in \mathfrak{L}(Y,H)$ and $\dim(Y)$ is finite, this implies that $\|\overline{B'S'(T-\cdot)C^*}\|_{\mathfrak{L}(Y,U)}$ is square integrable on $[0,T]$. Then for $y \in L_2(0,T;Y)$,

$$\begin{aligned}
&(\mathfrak{L}_\lambda^* y)(t) - (\mathfrak{L}^* y)x \\
&\qquad = \int_t^T \overline{B'S'(s-t)}\,(J_\lambda^* C^* - C^*)\,y(s)\,ds
\end{aligned}$$

where $J_\lambda^* = \lambda(\lambda I - A_H^*)^{-1}$, $\lambda \geq \lambda_0$. By (5.10), as $\lambda \longrightarrow \infty$

$$(5.11) \qquad \|\overline{B'S'(\cdot-t)}\,(J_\lambda^* C^* - C^*)\|_{L_2(t,T;U)} \leq b\|J_\lambda^* C^* - C^*\|_H \longrightarrow 0$$

and thus $(\mathfrak{L}_\lambda^* y) \longrightarrow (\mathfrak{L}^* y)(t)$ strongly for each $t \in [0,T]$. The desired result (Corollary 2.4) now follows from the dominated convergence theorem.

Next, we shall show the following theorem which replaces the results in Section 3 under the assumption (H5)--instead of (H3), (H2), $W = H$, and $V = D(A^*)'$.

Theorem 5.1. *The optimal solution* u^0 *to* (3.1) *is given by*

$$u^0_{t_0} = -B'\Pi(t)\,U(t,t_0)\,x \quad \text{for} \quad x \in H$$

$$\Pi(t)\,x = \int_t^T U^*(T,s)\,C^*CU(T,s)\,x\,ds\ , \quad x \in H$$

and suppose $K(t) = B'\Pi(t)$ *and* $L(t) = CU(T,t)$, $t \leqslant T$, *then* $K(\cdot) \in C_s(0,T;\mathfrak{L}(H,Y))$, *where* $U(t,s)$ *is jointly continuous on* $0 \leqslant s < t \leqslant T$ *in* H *and is defined by*

$$U(t,s)\,x = S(t-s)\,x - \int_s^t S(t-\sigma)\,BK(\sigma)\,U(\sigma,s)\,x\,d\sigma\ , \quad x \in H\ .$$

Moreover, $\|L(\cdot)B\|_{\mathfrak{L}(U,Y)}$ *is square integrable on* $[0,T]$ *and*

$$K(t)x = \int_t^T (L(s)\,B)^*L(s)\,x\,ds\ , \quad x \in H\ .$$

Proof: First note that if

$$u^\lambda = -(I + \mathfrak{L}_\lambda^*\mathfrak{L}_\lambda)^{-1}\mathfrak{L}_\lambda^* M_{t_0} x \quad \text{for} \quad x \in H\ ,$$

then for each $x \in H$, u^{λ} converges strongly to $u^0_{t_0}$ as $\lambda \longrightarrow \infty$ in $L_2(t_0,T;U)$ and the convergence is uniform in $t_0 \in [0,T]$ (see Lemma 3.1). Thus, using arguments similar to those given in the proof of Theorem 3.3, one can show that the self-adjoint operator $\Pi(t_0)$, $t_0 \leq T$ on H, defined by

$$\Pi(t_0)x = \int_{t_0}^{T} S^*(s-t_0)C^*\left[(I + \mathfrak{L}_{t_0}\mathfrak{L}^*_{t_0})^{-1}M_{t_0}x\right](s)\,ds\ , \qquad x \in H\ ,$$

satisfies

$$(5.12) \qquad \langle \Pi(t_0)x, x\rangle = \int_{t_0}^{T} \left\|CU(T,s)x\right\|_Y^2 ds\ , \qquad x \in H$$

where $(t,s) \longrightarrow U(t,s)x$, $x \in H$ is continuous and satisfy

$$(5.13) \qquad U(t,t_0)x = S(t-t_0)x - \int_{t_0}^{t} S(t-\sigma)Bu^0_{t_0}(\sigma)\,d\sigma$$

From (H5) and (5.10) one can show that

$$B'\Pi(t_0)x = \int_{t_0}^{T} \overline{B'S'(s-t_0)}C^*\left[(I + \mathfrak{L}_{t_0}\mathfrak{L}^*_{t_0})^{-1}M_{t_0}x\right](s)\,ds\ , \quad x \in H\ .$$

Since for $\Delta t \geq 0$, $B'S'(\cdot - (t_0 - \Delta t))C^* = B'S'(\cdot - t_0)S(\Delta t)C^*$ on $[t_0,T]$ and $t_0 \longrightarrow (I + \mathfrak{L}_{t_0}\mathfrak{L}^*_{t_0})^{-1}M_{t_0}x$ is strongly continuous for each $x \in H$, this implies that $K(\cdot) \in C_s(0,T;\mathfrak{L}(H,U))$.

$$(5.14) \qquad L(t_0)x = CS(T-t_0)x - \int_{t_0}^{T} S(T-s)B\left[(I + \mathfrak{L}^*_{t_0}\mathfrak{L}_{t_0})^{-1}\mathfrak{L}^*_{t_0}M_{t_0}x\right](s)\,ds$$

where

$$\mathcal{M}_{t_0}x = \mathcal{C}S(\cdot - t_0)x \in L_2(0,T;Y) , \quad x \in H .$$

Since dim(Y) is finite, say of dimension p ,

$$\overline{\mathcal{B}'S'(\cdot - t_0)\mathcal{C}^*y} = \Sigma\, y_i g_i(\cdot - t_0) , \quad y \in \mathbb{R}^p$$

where y_i is the ith component of y and $g_i(\cdot)$ is a U-valued square integrable function. Then, if e_i denotes the ith unit vector in $\mathbb{R}^p$,

$$e_i^T \mathcal{C} \int_{t_0}^{T} S(T-t)\mathcal{B}u(t)\,dt = \int_{t_0}^{T} \langle g_i(T-t), u(t)\rangle_U\, dt ,$$

and thus

$$\begin{aligned} e_i^T \mathcal{M}_{t_0}\mathcal{B}u &= e_i^T (\overline{\mathcal{B}'S'(\cdot - t_0)\mathcal{C}^*})^* u \\ &= \langle g_i(\cdot - t_0), u\rangle \quad \text{for} \quad u \in U . \end{aligned}$$

It then follows from (5.14) that $L(t)\mathcal{B}u$ is strongly measurable for each $u \in U$ and $\|L(\cdot)\mathcal{B}\|_{\mathfrak{L}(U,Y)}$ is square integrable on $[0,T]$.

Note that $u^\lambda = -\mathcal{B}'J_\lambda^*\Pi_\lambda(t)\mathcal{U}_\lambda(t,t_0)x$ for $x \in H$ (see (3.7)) where

$$\mathcal{B}'J_\lambda^*\Pi_\lambda(t_0)x = \int_{t_0}^{T} \overline{\mathcal{B}'S'(s-t_0)J_\lambda^*\mathcal{C}^*}\left[(I + \mathfrak{L}_\lambda\mathfrak{L}_\lambda^*)^{-1}\mathcal{M}_{t_0}x\right](s)\,ds$$

for $x \in H$. Combining (5.11) and the argument in the proof of Lemma 3.1

with the fact that u^{λ} converges strongly to $u^0_{t_0}$ in $L_2(t_0,T;U)$, we obtain

$$u^0_{t_0}(t) = -K(t)\,\mathcal{U}(t,t_0)\,x \quad \text{for} \quad x \in H \ .$$

The rest of the statements of Theorem 5.1 follow from (5.12), (5.13), and arguments similar to those given in Section 3. Q.E.D.

Corollary 5.2. *The functions* $t \longrightarrow K(t)x$ *for* $x \in H$ *and* $t \longrightarrow L(t)z$ *for* $z \in D_H(A)$ *are absolutely continuous on* $[0,T]$ *and they satisfy the Chandrasekhar equations* (3.23) *and* (3.24) *with* $x \in H$ *and* $z \in D_H(A)$.

We remark that the optimal quadratic problem for boundary controls of linear symmetric hyperbolic systems discussed in [23] can be formulated as above and thus Theorem 5.1 and Corollary 5.2 apply to such a problem. By duality, a similar result holds for the case when $H = V$ and $W = D_H(A)$, (H3) holds, and $\dim(U)$ is finite.

5.3 Parabolic Systems [5], [8], [17]

Consider the parabolic equation with Dirichlet boundary control:

$$(5.15)\qquad \begin{cases} \dfrac{\partial}{\partial t}\, y(t,\xi) = A_0 y(t,\xi)\,, & \xi \in \Omega \\ y(0) = y_0 & \\ y(t,\sigma) = u(t,\sigma)\,, & \sigma \in \Gamma \end{cases}$$

where A_0, Ω, and Γ are defined as in (5.8). If G is the Green map defined by (5.9), then (5.15) can be formulated as the evolution equation of (2.1):

$$\frac{d}{dt}\, x(t) = \mathcal{A}x(t) - \mathcal{A}Gu(t)$$

where $x(t) = y(t,\cdot) \in L^2(\Omega)$, $u(t) = u(t,\cdot) \in L^2(\Gamma)$ and $D(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega)$. It is known [17] that $\mathcal{A}$ generates an analytic semigroup $S(t)$ on H and that $Gu \in D((-\mathcal{A})^{\alpha})$, $0 \leqslant \alpha < 1/4$ where $(-\mathcal{A})^{\alpha}$ is the fractional operator of $-\mathcal{A}$ [20], [28].

Motivated by this example, we consider the following case [8]; $W = H$ and $V = D(\mathcal{A}^*)'$, $\mathcal{A}$ generates an analytic semigroup on H, and $\mathcal{B} = -\mathcal{A}G$ with $\mathrm{Range}(G) \subset D((-\mathcal{A})^{\alpha})$, $\alpha > 0$. In this case, (H2) and (H3) are not satisfied. However, by the closed graph theorem, $(-\mathcal{A})^{\alpha}G \longrightarrow \mathfrak{L}(U,H)$ and hence

$$\left\| S(t)\mathcal{B} \right\|_{\mathfrak{L}(U,H)} \leqslant \frac{M}{t^{1-\alpha}}\,, \qquad t \geqslant 0\,.$$

Thus, suppose $C \in \mathfrak{L}(H,Y)$, by Young's inequality $\mathfrak{L}_{t_0} \in \mathfrak{L}(L_2(t_0,T;U),L_2(t_0,T;Y))$ and the optimal $u^0_{t_0}$ is given by

$$u^0_{t_0} = -(I + \mathfrak{L}^*_{t_0}\mathfrak{L}_{t_0})^{-1}\mathfrak{L}^*_{t_0}M_{t_0}x , \quad x \in H .$$

Combining the arguments in [13] and those in [8], one can show that

$$u^0_{t_0} = -B'\Pi(t)\,U(t,t_0)\,x$$

and

(5.16) $$\langle \Pi(t_0)x,x\rangle_H = \int_{t_0}^T |CU(T,s)|^2 ds$$

where the evolution operator $U(\cdot,\cdot)$ is given by

(5.17) $$U(t,s)x = S(t-s)x - \int_s^t S(t-\sigma)BB'\Pi(\sigma)U(\sigma,s)x\,d\sigma , \quad x \in H .$$

Let $K(t)x = B'\Pi(t)$ for $x \in H$ and $t \leq T$. It then follows from Proposition 3.1 in [8] that $K(\cdot) \in C_s(0,T;\mathfrak{L}(H,U))$. Moreover, we have the following lemma.

Lemma 5.3. *There is a unique evolution operator of* (5.17) *satisfying*

(i) $(t,s) \longrightarrow U(t,s)$ *is continuous on* $0 \leq s \leq t \leq T$

(ii) $$\|U(t,s)B\| \leq \frac{M}{(t-s)^{1-\alpha}} , \quad t > s$$

Proof: Define a sequence of evolution operator $U_k(t,s)$ on $0 \leq s \leq t \leq T$ generated by

$$U_{k+1}(t,s) = S(t-s) - \int_s^t S(t-\sigma) BK(\sigma) U_k(\sigma,s)\, d\sigma$$

with $U_0(t,s) = 0$.

If $R_k(t,s) = U_k(t,s)(-A)^{1-\alpha}$ for $t > s$, then

$$R_{k+1}(t,s) = R_1(t,s) - \int_s^t S(t-\sigma) BK(\sigma) R_k(\sigma,s)\, x\, d\sigma .$$

By induction on k, one can show that

$$\|U_{k+1}(t,s) - U_k(t,s)\| \leq \frac{(c\Gamma(\alpha))^k}{\Gamma(k\alpha+1)} (t-s)^{k\alpha}, \quad k \geq 0 \tag{5.18}$$

and

$$\|R_k(t,s) - R_{k-1}(t,s)\| \leq \frac{(c\Gamma(\alpha))^k}{\Gamma(k\alpha)} (t-s)^{k\alpha-1}, \quad k \geq 1 \tag{5.19}$$

where

$$c = M \max_{0 \leq t \leq T} \|K(t)\|_{\mathfrak{L}(H,U)}$$

and $\Gamma(\cdot)$ is the classical gamma function. Here we used the well-known identity:

$$\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{\beta-1}d\sigma = (t-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} .$$

The estimate (5.18) implies that the sequence $U_k(t,s)$ converges in norm uniformly on $0 \leqslant s \leqslant t \leqslant T$ and thus $U(t,s) = \lim U_k(t,s)$ satisfies (5.17) and the statement (i). Suppose $U(t,s)$ and $\hat{U}(t,s)$ satisfy (5.17). Then we have

$$\|U(t,s) - \hat{U}(t,s)\| \leqslant \frac{c\Gamma(\alpha)}{\Gamma(\alpha+1)}(t-s)^\alpha \max_{s\leqslant\sigma\leqslant t} \|U(\sigma,s) - \hat{U}(\sigma,s)\| .$$

Hence, the uniqueness of solutions to (5.17) follows from the semigroup property of $U(\cdot,\cdot)$.

The estimate (5.19) implies that the sequence $R_k(t,s)$ converges uniformly in norm for $0 \leqslant s \leqslant t-\epsilon \leqslant T$ and every $\epsilon > 0$. As a consequence, $R(t,s) = \lim R_k(t,s)$, $t \geqslant s$ is uniformly continuous in $\mathfrak{L}(U,H)$ for $0 \leqslant s \leqslant t-\epsilon \leqslant T$ and every $\epsilon > 0$. Moreover,

$$\begin{aligned}\|R(t,s)\| &\leqslant \sum_{k=1}^{\infty} \Gamma(k\alpha)^{-1}(c\Gamma(\alpha))^k(t-s)^{k\alpha-1} \\ &\leqslant \sum_{k=1}^{\infty} \left[\Gamma(k\alpha)^{-1}(c\Gamma(\alpha))^k T^{\alpha(k-1)}\right](t-s)^{\alpha-1} \\ &\leqslant \tilde{M}(t-s)^{\alpha-1}\end{aligned}$$

For $x \in D((-A)^{1-\alpha})$ and $y \in H$,

$$\langle R(t,s)x,y\rangle_H = \langle(-A)^{1-\alpha}x, U^*(t,s)y\rangle_H , \quad t \geqslant s .$$

Since $(-A)^{1-\alpha}$ is closed, this implies that

$$(5.20) \qquad U^*(t,s)y \in D((-A^*)^{1-\alpha}) \quad \text{for } y \in H$$

and that

$$R(t,s) = \overline{U(t,s)(-A)^{1-\alpha}} .$$

Thus, the statement (ii) follows from the closed graph theorem. Q.E.D.

Now, from (5.16) and (5.20) arguments similar to those given in the proof of Theorem 3.6 yield

$$K(t)x = \int_t^T B^*L^*(s)x\,ds , \quad x \in H$$

where $L(t)x = CU(T,t)x$, $x \in H$ and $B^* = -G^*A^* = ((-A)^{\alpha}G)^*(-A^*)^{1-\alpha}$.

Lemma 5.4. *The evolution operator* $U(t,s)$ *defined by* (5.17) *satisfies*

$$U(t,s) = S(t-s) - \int_s^t U(t,\sigma)BK(\sigma)S(\sigma-s)\,d\sigma$$

on $0 \leqslant s \leqslant t \leqslant T$.

Proof: Define the evolution operator V by

$$V(t,s) = S(t-s) - \int_s^t U(t,\sigma)BK(\sigma)S(\sigma-s)\,d\sigma$$

for $0 \leq s \leq t \leq T$. By (ii) of Lemma 5.3, $(t,s) \longrightarrow V(t,s)$ is continuous and from (5.17)

$$V(t,s) = S(t-s) - \int_s^t \left[S(t-\sigma) - \int_\sigma^t S(t-\tau)BK(\tau)\,U(\tau,\sigma)\,d\tau\right] BK(\sigma)\,S(\sigma-s)\,d\sigma$$

where

$$\int_s^t \left[\int_\sigma^t S(t-\tau)BK(\tau)\,U(\tau,\sigma)\,d\tau\right] BK(\sigma)\,S(\sigma-s)\,d\sigma$$
$$= \int_s^t S(t-\tau)BK(\tau)\int_s^T U(\tau,\sigma)\,BK(\sigma)\,S(\sigma-s)\,d\sigma\,d\tau \ .$$

Thus, we obtain

$$V(t,s) = S(t-s) - \int_s^t S(t-\tau)BK(\tau)\left[S(\tau-s) - \int_s^t U(\tau,\sigma)\,BK(\sigma)\,S(\sigma-s)\,d\sigma\right]d\tau$$
$$= S(t-s) - \int_s^t S(t-\tau)\,BK(\tau)\,V(\tau,s)\,d\tau \ .$$

Since the solution of (5.17) is unique, this implies that $U(t,s) = V(t,s)$ on $0 \leq s \leq t \leq T$. Q.E.D.

From Lemma 5.4, $L(t)$, $t \leq T$ satisfies

$$L(t)x = CS(T-t)x - \int_t^T L(s)BK(s)S(s-t)x \ , \quad x \in H \ .$$

Note that $L(t)B = CU(T,t)B$, $t < T$. Thus, for $x \in D(A)$, $t \longrightarrow L(t)x$ is continuously differentiable on $[0,T)$ and satisfies

$$\frac{d}{dt} L(t)x = -L(t)(A - BK(t))x \ , \ x \in D(A) .$$

Hence, we obtain (compare it with the result in Sorine [26]).

Theorem 5.5. *The operators* $K(\cdot) \in C_s(0,T;\mathfrak{L}(H,U))$ *and* $L(\cdot) \in C_s(0,T;\mathfrak{L}(H,Y))$ *satisfy the equations* (3.23) *and* (3.24) *in which* $t \longrightarrow K(t)x$, $x \in H$ *and* $t \longrightarrow L(t)z$, $z \in D(A)$ *are continuously differentiable on* $[0,T]$.

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16. Abstract

A set of equations known as Chandrasekhar equations arising in the linear quadratic optimal control problem is considered. In this paper, we consider the linear time-invariant system defined in Hilbert spaces involving unbounded input and output operators. For a general class of such systems, we derive the Chandrasekhar equations and establish the existence, uniqueness, and regularity results of their solutions.

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